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SIMPLE WAVES IN NONCONDUCTING MAGNETIZABLE MEDIUM

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The propagation of small amplitude waves through a nonconducting, isotropically magnetizable medium is studied, and simple wave equations obtained. Simple waves in an ideal magnetizable gas are studied in detail. The problem of stability is considered for the ideal gas and a magnetizable fluid, and the parameter values for which the wave phase velocities become imaginary are determined.

The motion of a medium which does not conduct current but can be isotropically and nonuniformly magnetized in an external magnetic field, can be described by the following system of equations [1]

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} = 0, \qquad \rho T \frac{d}{dt}(s + s^*) = \tau_{ik} \frac{\partial v_i}{\partial x_k} + \lambda^\circ \Delta T \tag{1}$$

$$\rho \frac{d\mathbf{v}}{dt} + \nabla (p + \psi) - M \nabla H = \eta_1 \Delta \mathbf{v} + \left(\eta_2 + \frac{1}{3}\eta_1\right) \nabla \operatorname{div} \mathbf{v}$$

$$\operatorname{div} \mathbf{B} = 0, \qquad \operatorname{div} \mathbf{E} = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = -c \operatorname{rot} \mathbf{E}, \qquad \varepsilon \frac{\partial \mathbf{E}}{\partial t} = c \operatorname{rot} \mathbf{H}$$

$$\mathbf{B} = \mathbf{H} + 4\pi M(\rho, T, H)\mathbf{H}/H, \qquad p = p(\rho, s), \qquad T = T(\rho, s)$$

$$\left(\psi = \int_0^H \left\{M - \rho \left(\frac{\partial M}{\partial \rho}\right)_{T, H}\right\} dH, \qquad s^* = \frac{1}{\rho}\int_0^H \left(\frac{\partial M}{\partial T}\right)_{\rho, H} dH\right)$$

Here τ_{ik} is the viscous stress tensor; λ°, η_1 and η_2 are constant coefficients of heat conductivity, first and second viscosity, respectively; $M(\rho, T, H) \equiv (4\pi)^{-1}(\mu - 1) H$ is a function of magnetization (assumed known), $\mu = \mu(\rho, T, H)$ is the magnetic permeability of the medium, the dielectric permeability ε is constant and free charges are absent.

The propagation of small amplitude waves in such a medium can be described by the following system of seven equations:

$$\frac{\partial u_i}{\partial t} + x_{ik} \frac{\partial u_k}{\partial x} = d_{ik} \frac{\partial^2 u_k}{\partial x^2} \qquad (i, k = 1, 2, \ldots 7) \tag{2}$$

$$u_1 \equiv \rho', \; u_2 \equiv s', \; u_3 \equiv v_x', \; u_4 \equiv B_y', \; u_5 \equiv B_z', \; u_6 \equiv E'_y, \; u_7 \equiv E_z'$$

Here u_i denote perturbations of the variables and the matrices $\|x_{ik}\|$ and $\|d_{ik}\|$ have the following nonzero components:

$$x_{13} = \rho, \quad x_{23} = \rho N\,[m\mu_T B^2 (\mu_\rho + \mu_T T_\rho) - s_\rho{}^* - s_T^* T_\rho] \tag{3}$$
$$x_{26} = -cm\mu\mu_T N B_z, \qquad x_{27} = -x_{26} B_y / B_z$$
$$x_{31} = (p_\rho + \psi_\rho + \psi_T T_\rho) / \rho + m\rho\mu_\rho B^2 (\mu_\rho + \mu_T T_\rho)$$
$$x_{32} = (p_s + \psi_T T_s) / \rho + m\rho\mu_\rho B^2 \mu_T T_s$$
$$x_{34} = -m\mu\mu_\rho \rho B_y, \qquad x_{35} = x_{34} B_z / B_y, \qquad x_{47} = -x_{56} = -c$$
$$x_{61} = -4\pi\rho\mu\varepsilon^{-1} mc\,(\mu_\rho + \mu_T T_\rho)\, B_z$$
$$x_{62} = -4\pi\rho\mu\varepsilon^{-1} mc\mu_T T_s B_z, \qquad x_{64} = -4\pi\rho mc\mu_H \varepsilon^{-1} B^{-1} B_y B_z$$
$$x_{65} = 4\pi\rho mc\varepsilon^{-1}\,[\mu^2 + \mu_H B^{-1} (B_x{}^2 + B_y{}^2)]$$
$$x_{71} = -x_{61} B_y / B_z, \qquad x_{72} = -x_{62} B_y / B_z$$
$$x_{74} = -4\pi\rho mc\varepsilon^{-1}\,[\mu^2 + \mu_H B^{-1}(B_x{}^2 + B_z{}^2)], \quad x_{75} = -x_{64}$$
$$d_{21} = N\lambda^\circ T_\rho / \rho T, \quad d_{22} = N\lambda^\circ T_s / \rho T, \quad d_{33} = (\eta_2 + {}^4/_3\eta_1)/\rho$$
$$(m^{-1} = 4\pi\rho\mu\,(\mu^2 + \mu_H B), \; N^{-1} = 1 + T_s\,(s_T{}^* - m\mu_T{}^2 B^2))$$

The four scalar variables v'_y, v_z', B_x' and E_x' of the system (1) satisfy the equations

$$\frac{\partial E_x'}{\partial x} = \frac{\partial E_x'}{\partial t} = 0, \qquad \frac{\partial B_x'}{\partial x} = \frac{\partial B_x'}{\partial t} = 0 \tag{4}$$
$$\rho \frac{\partial v_y'}{\partial t} = \eta_1 \frac{\partial^2 v_y'}{\partial x^2}, \qquad \rho \frac{\partial v_z'}{\partial t} = \eta_1 \frac{\partial^2 v_z'}{\partial x^2}$$

from which it follows that $B_x' = \text{const}$, $E_x' = \text{const}$, the perturbations v_x' and v_z' (which do not appear in (2)) in a nondissipative system $(d_{ik} = 0)$ do not vary in the plane wave and in the general case their initial values diffuse into the medium independently of the wave propagation. The field perturbation $\mathbf{H}'$ is eliminated from (2) using the linear relation

$$(4\pi\rho m)^{-1}\,\mathbf{H}' = \mathbf{l}_1\rho' + \mathbf{l}_2 s' + \mathbf{l}_4 B_y' + \mathbf{l}_5 B_z' \tag{5}$$
$$\mathbf{l}_1 = -\mathbf{B}\,(\mu_\rho + \mu_T T_\rho)\,\mu, \qquad \mathbf{l}_2 = -\mathbf{B} T_s \mu$$
$$\mathbf{l}_4 = (-\mu_H B_x B_y B^{-1}, \qquad \mu^2 + \mu_H B^{-1}(B_x{}^2 + B_z{}^2), \quad -\mu_H B_y B_z B^{-1})$$
$$\mathbf{l}_5 = (-\mu_H B_x B_z B^{-1}, \qquad -\mu_H B_y B_z B^{-1}, \quad \mu^2 + \mu_H B^{-1}(B_x{}^2 + B_y{}^2))$$

Equations (2), (4) and (5) are obtained from (1) by linearization with respect to the unperturbed state of the medium, under small perturbations u_i depending only on x and t. The unperturbed state of the medium is determined by the constant values of the parameters $\mathrm{v} = 0$, ρ, p, s, T, $\mathbf{B}$, $\mathbf{E}$ and the derivatives

$$p_\rho \equiv \frac{\partial p}{\partial \rho}, \quad p_s \equiv \frac{\partial p}{\partial s}, \quad T_\rho \equiv \frac{\partial T}{\partial \rho}, \quad T_s \equiv \frac{\partial T}{\partial s}, \quad \mu_H \equiv \frac{\partial \mu}{\partial H},$$
$$\mu_T \equiv \frac{\partial \mu}{\partial T}, \quad \mu_\rho \equiv \frac{\partial \mu}{\partial \rho}$$

In the following we shall limit ourselves to considering a nondissipative, nonconducting medium, i.e. we shall assume $d_{ik} = 0$. (Note that in [2] the medium for which the propagation of the sound waves was dealt with in example, was erroneously called nonconducting). Seeking a solution of (2) in the form $u_m = u_m{}^\circ \exp\,(ikx - i\omega t)$, we obtain for the phase velocities of the waves $\lambda \equiv \omega / k$, a characteristic equation of the form

$$\lambda\,[\lambda^6 - \lambda^4 (A_1 + c^2 A_2) + \lambda^2 c^2 (A_3 + c^2 A_4) + c^4 A_5] = 0 \tag{6}$$

in which the coefficients A_i are independent of c. The expressions for these coefficients written in terms of the matrix elements x_{ik} are sufficiently awkward and are not given here. Determining the roots of this equation in the form of the following expansion:

$$\lambda^2 = a_m c^{2m} + a_{m-1} c^{2(m-1)} + \ldots + a_0 + a_{-1} c^{-2} + \ldots$$

we can show that only two values of m are possible, $m = 0$ and $m = 1$. Discarding terms of the order c^{-2}, we obtain the roots of the characteristic equation (6) in the form

$$\lambda_1 = 0, \qquad \lambda_{2,3} = \pm c / \sqrt{\mu\varepsilon} \tag{7}$$

$$\lambda_{4,5} = \pm \frac{c}{\sqrt{\mu\varepsilon}} \left(1 + L_2 \frac{B_z^2 + B_y^2}{B^2}\right)^{1/2}$$

$$\lambda_{6,7} = \pm \left[\rho x_{31} + x_{23} x_{32} - L_1 \frac{B_z^2 + B_y^2}{B^2} \left(1 + L_2 \frac{B_z^2 + B_y^2}{B^2}\right)^{-1}\right]^{1/2}$$

$$L_1 = 4\pi\rho\mu^3 m^2 B^2 (\rho\mu_\rho + N\mu_T x_{32}) [\rho (\mu_\rho + \mu_T T_\rho) + \mu_T T_s x_{23}]$$

$$L_2 = (m\mu^2 \mu_T^2 T_s N B^2 - \mu_H B)(\mu^2 + \mu_H B)^{-1}$$

Seven roots of (6) given in (7) determine three distinct types of waves:

1) entropic wave with phase velocity λ_1,

2) first electromagnetic wave with phase velocity $\lambda_{2,3}$ and second electromagnetic wave with phase velocity $\lambda_{4,5}$,

3) magnetohydrodynamic wave with phase velocity $\lambda_{6,7}$.

The second electromagnetic and the magnetohydrodynamic wave are anisotropic, i.e. the velocity of propagation of their wavefronts depends on their orientation with respect to the field $\mathbf{B}$.

As we know [3], the simple waves are closely related to the small amplitude waves. In the case under consideration, the differential equations for the simple waves can be written in the form

$$\frac{du_1}{r_1^{(\lambda)}} = \frac{du_2}{r_2^{(\lambda)}} = \ldots = \frac{du_7}{r_7^{(\lambda)}} \tag{8}$$

Here $r_1^{(\lambda)}, r_2^{(\lambda)}, \ldots, r_7^{(\lambda)}$ are the components of the right eigenvector of the matrix $\|x_{ik}\|$ corresponding to the given phase velocity λ. The expressions for these components which are proportional to the amplitudes of the corresponding waves, in the present case have the form

$$\begin{aligned}
r_1 &= \lambda^6 - \lambda^4 (x_{23} x_{32} + c^2 R_{11}) + \lambda^2 c^2 (R_{12} + c^2 R_{13}) - c^4 R_{14} \\
r_2 &= \lambda^4 (x_{23} x_{31} + c^2 R_{21}) + \lambda^2 c^2 (R_{22} + c^2 R_{23}) + c^4 R_{24} \\
r_3 &= \lambda^5 x_{31} - \lambda^3 c^2 R_{31} + \lambda c^4 R_{32} \\
r_4 &= -\lambda^4 c x_{71} - \lambda^2 c^2 (R_{41} + c^2 R_{42}) - c^4 R_{43} \\
r_5 &= \lambda^4 c x_{61} - \lambda^2 c^2 (R_{51} + c^2 R_{52}) - c^4 R_{53} \\
r_6 &= \lambda^5 x_{61} - \lambda^3 c (R_{61} + c^2 R_{62}) + \lambda c^3 R_{63} \\
r_7 &= -\lambda^5 x_{71} + \lambda^3 c (R_{71} + c^2 R_{72}) + \lambda c^3 R_{73}
\end{aligned} \tag{9}$$

The quantities R_{ik} which shall be needed below, have the following values:

$$R_{11} = 4\pi\rho m\varepsilon^{-1}[\mu^2 + \mu_H B_x^2 B^{-1} + m\mu^2\mu_T^2 N T_s (B^2_y + B_z^2)] + (\mu\varepsilon)^{-2} \quad (10)$$
$$R_{13} = (\mu\varepsilon)^{-1} R_{11} - (\mu\varepsilon)^{-2}$$
$$R_{14} = 4\pi\rho m\mu^{-1}\varepsilon^{-2}x_{23}[x_{32}(\mu^2 + \mu_H B_x^2 B^{-1}) - m\mu^2\mu_\rho\mu_T \rho T_s (B_y^2 + B_z^2)]$$
$$R_{21} = 4\pi\rho m^2\mu^2\mu_T N\varepsilon^{-1}(\mu_\rho + \mu_T T_\rho)(B_z^2 + B_y^2) = -\varepsilon\mu R_{23}$$
$$R_{24} = 4\pi\rho m\mu^{-1}\varepsilon^{-2}x_{23}[x_{31}(\mu^2 + \mu_H B_x^2 B^{-1}) - m\mu^2\mu_\rho\rho(\mu_\rho + \mu_T T_\rho)(B^2_z + B_y^2)]$$
$$R_{32} = x_{31}[1 + L_2(B_y^2 + B_z^2)B^{-2}]\mu^{-2}\varepsilon^{-2} - 4\pi\rho\mu m^2\varepsilon^{-2}(\mu_\rho + \mu_T T_\rho)(\rho\mu_\rho + \mu_T x_{32} N)(B_z^2 + B_y^2)$$
$$R_{42} = -4\pi\rho m\varepsilon^{-2}B_y(\mu_\rho + \mu_T T_\rho)$$
$$R_{43} = 4\pi\rho m\varepsilon^{-2}B_y x_{23}[x_{32}(\mu_\rho + \mu_T T_\rho) - \mu_T T_s x_{31}]$$
$$R_{52} = R_{42}B_z B_y^{-1}, \quad R_{53} = R_{43}B_z B_y^{-1}$$
$$R_{62} = R_{52}, \quad R_{72} = R_{42}$$

Proceeding from (9) we can obtain, for each value of the phase velocity λ, the corresponding differential equations of the simple waves, discarding in $r_i^{(\lambda)}$ the terms of the order of c^{-2} and higher.

Equations for a simple entropic wave ($\lambda = 0$) have the form

$$\frac{d\rho}{-R_{14}} = \frac{ds}{R_{24}} = \frac{dv_x}{0} = \frac{dB_y}{-R_{43}} = \frac{dB_z}{-R_{43}B_z B_y^{-1}} = \frac{dE_y}{0} = \frac{dE_z}{0} \quad (11)$$

From this it follows that in a simple entropic wave the velocity and the electric field do not vary. In contrast to the entropic wave in a nonmagnetic medium, here the field $\mathbf{B}$ varies, but the wave is plane polarized since from (11) it follows that $B_z B_y^{-1} =$ const. Choosing a coordinate system in which $B_z = 0$, from (11) we obtain

$$\frac{ds}{d\rho} = -\frac{R_{24}}{R_{14}}, \qquad \frac{dB_y}{d\rho} = \frac{R_{43}}{R_{14}}, \qquad v_x, E_y, E_z = \text{const} \quad (12)$$

Equations determining the second electromagnetic simple wave in a magnetizable medium ($\lambda = \lambda_{4,5}$) assume, after computing the corresponding coefficients, the following form:

$$\frac{d\rho}{0} = \frac{ds}{m\mu\mu_T N B_y^2} = \frac{dv_x}{0} = \frac{dB_y}{-B_y} = \frac{dB_z}{-B_z} = \frac{dE_y}{-\lambda c^{-1}B_z} = \frac{dE_z}{\lambda c^{-1}B_y} \quad (13)$$

Thus in the second simple electromagnetic wave the density and the velocity of the medium do not vary. This wave is also plane polarized. Then, setting $B_z = 0$, from (13) we obtain

$$\frac{dE_z}{dB_y} = -\frac{\lambda_{4,5}}{c}, \qquad \frac{ds}{dB_y} = -m\mu\mu_T N B_y, \qquad \rho, v_x, E_y = \text{const} \quad (14)$$

In the first electromagnetic wave ($\lambda = \lambda_{2,3}$) only the electric and magnetic fields vary, and the wave is also plane polarized.

Finally we consider simple magnetohydrodynamic waves in a magnetizable medium. By virtue of (7), (9) and (10) we obtain from (8) for $\lambda = \lambda_{6,7}$

$$\frac{d\rho}{\rho R_{32}} = \frac{ds}{\lambda^2 R_{23} + R_{24}} = \frac{dv_x}{\lambda R_{32}} = \frac{dB_y}{-(\lambda^2 R_{42} + R_{43})} = \frac{dB_z}{-B_z B_y^{-1}(\lambda^2 R_{42} + R_{43})} = \frac{dE_y}{0} = \frac{dE_z}{0} \quad (15)$$

Thus in a simple magnetohydrodynamic wave only the electric field does not vary and, similarly to all previous waves, this wave is also plane polarized. Setting $B_z = 0$ from (15) we now obtain

$$\frac{dv_x}{d\rho} = \frac{\lambda_{6,7}}{\rho}, \qquad \frac{ds}{d\rho} = -\frac{R_2}{R_1}, \qquad E_y, E_z = \text{const}, \tag{16}$$

$$\frac{dB_y}{d\rho} = 4\pi m B_y \mu^2 \left(1 + L_2 \frac{B_y{}^2}{B^2}\right)^{-1} (\mu_T T_s x_{23} + \rho (\mu_\rho + \mu_T T_\rho))$$

$$R_1 = \rho x_{31} \mu^{-2} \varepsilon^{-2} (1 + L_2 B_y{}^2 B^{-2}) - 4\pi\rho\mu\varepsilon^{-2} m^2 (\mu_\rho + \mu_T T_\rho) \times (\rho\mu_\rho + \mu_T N x_{32}) B_y{}^2$$

$$R_2 = 4\pi\rho m \mu^{-1} \varepsilon^{-2} [m\mu^2 (\mu_\rho + \mu_T T_\rho)(x_{23}\rho\mu_\rho + \lambda^2 \mu_T N) B_y{}^2 - x_{23} x_{31} (\mu^2 + \mu_H B^2{}_x B^{-1})]$$

All these types of simple waves exist, provided that the phase velocities are real. It follows from (7) that in a magnetizable medium the quantities $\lambda_{4,5}$ and $\lambda_{6,7}$ may become purely imaginary at certain values of the parameters. The system (2) then ceases to be hyperbolic and the medium becomes unstable. An instability of a similar type has already been observed in the magnetic gas dynamics when investigating waves in a plasma with anisotropic pressure [4] and in the analysis of sound waves in a magnetizable conducting medium [2].

We shall assume that a magnetizable nonconducting medium becomes unstable and its motion can no longer be described by the system (1), if $\vartheta = \arcsin B_y B^{-1}$ is found such, that either $\lambda_{4,5}$ or $\lambda_{6,7}$ becomes imaginary. This may occur in one of the following three cases:

$$1)\ L_2 + 1 < 0, \quad 2)\ \rho x_{31} + x_{23} x_{32} < 0, \quad 3)\ \rho x_{31} + x_{23} x_{32} > 0 \tag{17}$$
$$L_2 + 1 \lessgtr 0 \qquad (1 + L_2)(\rho x_{31} + x_{23} x_{32}) - L_1 \gtrless 0$$

In the first case the second electromagnetic wave is unstable, and in the remaining cases it is the magnetohydrodynamic wave. We note that none of the above cases can occur in a nonmagnetic medium.

Let us now consider some models of the magnetizable media. The state of an ideal gas in weak magnetic fields can be described by the following equations (R is the gas constant and $\varkappa = c_p / c_v$ is the ratio of specific heats):

$$p = \rho^\varkappa \exp(s / c_v), \quad T = R^{-1} \rho^{\varkappa - 1} \exp(s / c_v) \tag{18}$$
$$\mathbf{B} = \mu(\rho, T)\mathbf{H}, \quad T(\mu - 1) / \mu\rho = \text{const}$$

In this case

$$L_1 = a^2 \frac{\alpha(\mu - 1)(2 - \varkappa + \varkappa\alpha)^2}{(\varkappa - 1)(1 + \varkappa\alpha)^2}, \qquad L_2 = \frac{\varkappa\alpha(\mu - 1)}{1 + \varkappa\alpha}$$
$$(\alpha \equiv (\mu - 1)(\varkappa - 1) B^2 (4\pi\rho\mu a^2)^{-1}, \ a^2 \equiv p_\rho = \varkappa R T)$$

and setting $B_z = 0$, we obtain from (7)

$$\lambda_{4,5} = \pm \frac{c}{\sqrt{\mu\varepsilon}} \left(1 + \frac{\varkappa\alpha(\mu - 1)}{1 + \varkappa\alpha} \sin^2\vartheta\right)^{1/2}, \qquad \lambda_{6,7} = \pm a \delta_0{}^{1/2} \left(\frac{\delta_1 + \sin^2\vartheta}{\delta_2 + \sin^2\vartheta}\right)^{1/2} \tag{19}$$

$$\delta_0 \equiv \frac{3\varkappa - 4 - \varkappa\alpha}{\varkappa(\varkappa - 1)}, \qquad \delta_1 \equiv \frac{(\varkappa - 1)(1 + \alpha)}{\alpha(\mu - 1)(3\varkappa - 4 - \varkappa\alpha)}, \qquad \delta_2 = \frac{1 + \varkappa\alpha}{\varkappa\alpha(\mu - 1)}$$

Analysing these expressions we find that an ideal magnetizable gas with $\mu > \varkappa^{-1}$ and $2 > \varkappa > {}^4/_3$ is stable only within the following range of values of the dimensionless parameter α:

$$-\varkappa^{-1} < \alpha < \alpha^*$$

$$\alpha^* \equiv \frac{1}{2\varkappa}\left(3\varkappa + 4 - \frac{\varkappa - 1}{\mu - 1}\right) + \sqrt{\frac{1}{4\varkappa^2}\left(3\varkappa - 4 + \frac{\varkappa - 1}{\mu - 1}\right)^2 + \frac{\varkappa - 1}{\varkappa(\mu - 1}}$$

Thus a diamagnetic ($\mu < 1$) and paramagnetic ($\mu > 1$) substance should be unstable in the above sense in the fields

$$B > \sqrt{\frac{4\pi\mu p}{(\varkappa - 1)(1 - \mu)}} \quad (\mu < 1), \qquad B > \sqrt{\frac{4\pi\mu\varkappa\alpha^* p}{(\varkappa - 1)(\mu - 1)}} \quad (\mu > 1) \tag{20}$$

Under the usual conditions ($p \sim 10^6$ dyn/cm ^{2}) the stability can be expected to be disturbed at sufficiently large values of $|\mu - 1|$, as the magnitude of the field $\mathbf{B}$ is restricted by the assumption of linearity between $\mathbf{B}$ and $\mathbf{H}$. If, on the other hand, we assume that the Clausius-Mosotti law of magnetization holds also in the strong fields, then with the values of $|\mu - 1| \sim 10^{-6}$, instability may occur in accordance with (20) in the fields of the order of 100 teslas.

Let us consider the simple wave equations for the ideal gas, described by (18).

From (10), (11) and (18) we obtain the following expressions for the entropic wave:

$$\frac{ds}{d\rho} = -\frac{\varkappa R}{(\varkappa - 1)\rho}\left(1 - \frac{\alpha(\mu - 1)(2 - \varkappa)B_y^2}{(\varkappa - 1)B^2}\right)\left(1 + \frac{\varkappa\alpha(\mu - 1)B_y^2}{(\varkappa - 1)B^2}\right)^{-1} \tag{21}$$

$$\frac{dB_y}{d\rho} = \frac{2B_y(\mu - 1)}{\rho}\left(1 + \frac{\varkappa\alpha(\mu - 1)B_y^2}{(\varkappa - 1)B^2}\right)^{-1}$$

Taking into account the fact that (18) implies

$$\frac{T}{T_0} = \left(\frac{\rho}{\rho_0}\right)^{\varkappa - 1}\exp\left(-\frac{s_0 - s}{c_v}\right), \quad \frac{\mu - 1}{\mu} = \frac{\mu_0 - 1}{\mu_0}\left(\frac{\rho}{\rho_0}\right)^{2 - \varkappa}\exp\left(\frac{s_0 - s}{c_v}\right) \tag{22}$$

$$\alpha = \alpha_0\left(\frac{B}{B_0}\right)^2\left(\frac{\rho}{\rho_0}\right)^{2(1 - \varkappa)}\exp\left(\frac{2(s_0 - s)}{c_v}\right) \quad \left(\alpha_0 \equiv \frac{(\mu_0 - 1)(\varkappa - 1)B^2}{4\pi\rho_0\mu_0 a_0^2}\right)$$

consequently, in order to obtain simple wave equations in the finite form we must integrate a nonlinear system (21), which is rather complicated. Although certain general conclusions concerning integral curves can be drawn from (21), we shall only consider the small values of $|\mu_0 - 1|$. Discarding in (21) terms of the order of $(\mu_0 - 1)^2$ and higher, we obtain after integration

$$s - s_0 = \varkappa c_v \ln(\rho_0 / \rho), \quad B_y - B_{y0} = B_{y0}(\mu_0 - 1)[(\rho / \rho_0)^2 - 1]$$

The first equation expresses the fact that the pressure in the entropic wave does not vary (with the accuracy of up to $(\mu_0 - 1)^2$), while the second equation determines the variation in the magnetic induction as function of density.

For the second electromagnetic wave from (7) and (19) we have

$$\frac{ds}{dB_y} = \frac{B_y(\mu - 1)}{4\pi\rho\mu T(1 + \varkappa\alpha)}, \qquad \frac{dE_z}{dB_y} = \pm\frac{1}{\sqrt{\mu\varepsilon}}\left(1 + \frac{\varkappa\alpha(\mu - 1)B_y^2}{(1 + \varkappa\alpha)B^2}\right)^{1/2} \tag{23}$$

Let Λ be the velocity of wave propagation relative to the coordinate system, chosen so that $\Lambda = v_x + \lambda$. Since ρ and v_x in the electromagnetic simple waves do not vary, we use (7), (13) and (18) to obtain

$$\frac{d\Lambda_{4,5}}{dB_y} = \frac{3cB_y \varkappa\alpha(\mu-1)}{2\sqrt{\varepsilon}\,\lambda_{4,5}\mu B^2(1+\varkappa\alpha)}\left(1-\frac{\varkappa\alpha(5+3\varkappa\alpha)B_y^2}{3(1+\varkappa\alpha)^2B^2}\right)$$

This shows that for $\varkappa\alpha > -1$ the velocity of propagation of these waves increases with B_y, with consequent "overlapping" and jumps in the values of B_y, s and E_z.

Equations (23) can be integrated. In fact, substituting (22) into the first equation of (23) and integrating, we obtain

$$B^2 = B_{x0}^2 + B_y^2 = B_0^2\left(1-\frac{2(s_0-s)}{\varkappa\alpha_0 c_v}\right)\exp\left(-\frac{2(s_0-s)}{c_v}\right)$$

The second equation of (23) now yields

$$E_z - E_{z0} = \pm\int_{B_{y0}}^{B_y}\left(1+\frac{\varkappa\alpha(\mu-1)B_y^2}{(1+\varkappa\alpha)B^2}\right)^{1/2}\frac{dB_y}{\sqrt{\mu\varepsilon}}$$

The simple electromagnetic waves were studied for the case $\varepsilon = \varepsilon(E)$, $\mu = \mu(H)$ in [5]. Equations of this type can be obtained from (13) by setting $\mu_T = 0$.

Simple magnetohydrodynamic wave equations in an ideal gas have, in accordance with (15) and (19), the form

$$\frac{dv_x}{d\rho} = \pm\frac{a}{\rho}\delta_0^{1/2}\left(\frac{\delta_1+B_y^2B^{-2}}{\delta_2+B_y^2B^{-2}}\right)^{1/2}, \quad \frac{dB_y}{d\rho} = \frac{B_y(\mu-1)(\alpha\varkappa+2-\varkappa)}{\rho[1+\alpha\varkappa+(\mu-1)\varkappa\alpha B_y^2B^{-2}]} \tag{24}$$

$$\frac{ds}{d\rho} = \frac{\alpha\varkappa R}{\rho}\,\frac{(\mu-1)(2-\varkappa)(\alpha+\lambda_{6,7}\bar{a}^{-2})B_y^2 - B^2(\varkappa-1)}{B^2(\varkappa-1)(1+\varkappa\alpha)+\alpha(\mu-1)B_y^2[\alpha\varkappa(\varkappa-2)+3\varkappa-4]}$$

From this it follows that in a paramagnetic medium we always have $dB_y / d\rho > 0$, so that the increase in the density of the medium is accompanied by an increase in the field strength. Discarding terms of the order of $(\mu_0 - 1)^2$ and higher we obtain on integrating (24)

$$v_x - v_{x0} = \frac{2a_0}{\varkappa-1}\left[\left(\frac{\rho}{\rho_0}\right)^{(\varkappa-1)/2} - 1 + \frac{\alpha_0(\varkappa-1)}{6}\left(\left(\frac{\rho}{\rho_0}\right)^{3(1-\varkappa)/2}-1\right)\right]$$

$$B_y - B_{y0} = B_{y0}(\mu_0-1)\left[\left(\frac{\rho}{\rho_0}\right)^{2-\varkappa}-1\right]$$

$$s - s_0 = \frac{\varkappa R\alpha_0}{2(\varkappa-1)}\left[\left(\frac{\rho_0}{\rho}\right)^{2(\varkappa-1)}-1\right]$$

Moreover, since

$$\frac{d\Lambda_{6,7}}{d\rho} = \frac{a_0(\varkappa+1)}{2\rho_0}\left(\frac{\rho}{\rho_0}\right)^{(\varkappa-3)/2}\left[1-\frac{\alpha_0(\varkappa-1)(5-3\varkappa)}{2(\varkappa+1)}\left(\frac{\rho_0}{\rho}\right)^{2(\varkappa-1)}\right]$$

then in a paramagnetic medium $(\alpha_0 > 0)$ the difference in the velocity of propagation of the points with different densities is smaller than in a nonmagnetic gas, and this causes a delay in transforming the simple waves into shocks. On the contrary, in a diamagnetic medium the formation of shocks proceeds rapidly.

Considering an ideal gas in strong magnetic fields, when it is magnetically saturated, we can assume that

$$M = \rho K(\theta - T) \quad (\theta > T,\ K = \text{const}) \tag{25}$$

In this case we have

$$L_1 = \mu\beta \frac{a^2(\tau - \varkappa)^2}{\varkappa(\varkappa - 1)(1 - \beta)^2}, \qquad L_2 = \frac{\mu}{1 - \beta} - 1$$

$$(\tau \equiv \theta T^{-1} > 1, \qquad \beta \equiv 4\pi\rho K^2 T c_v^{-1})$$

Analysis of the expressions for $\lambda_{4,5}$ and $\lambda_{6,7}$ in the case of a saturated ideal gas shows, that instability occurs when $\beta > 1$. We note that the dimensionless parameter β is independent of the magnetic field and the condition $\beta > 1$ can be written in the form

$$M > a(\tau - 1)\rho^{1/2}(4\pi\varkappa(\varkappa - 1))^{-1/2}$$

The simple wave equations in a saturated gas are basically the same as those discussed before, and can be written using (25), (3) and (8) – (10).

Finally, we consider a magnetizable medium with properties resembling those of ferromagnetic fluids [6]. We assume that the state of the medium in the absence of an electromagnetic field is determined by

$$T = T(s, \rho(p, T)) = T^\circ(s, p) \tag{26}$$

with the following known parameters obtained experimentally: isentropic speed of sound $a_s (\equiv \sqrt{p_\rho})$, the coefficient of thermal expansion $\gamma_1 \equiv -\rho_T / \rho$, the coefficient of isothermal compression $\gamma_2 \equiv \rho_p / \rho$ and the specific heat at constant pressure $c_p \equiv T / T_s^\circ$. Then T_ρ and T_s in (3) must be replaced by the expressions based on (26), namely $T_\rho = T_p^\circ / \rho\gamma_2$, $T_s = T(1 - T_\rho\rho_T)c_p^{-1} = T(1 + T_p^\circ\gamma_1\gamma_2^{-1})c_p^{-1}$

$$(T_s^\circ = T_s + T_\rho\rho_T T_s^\circ)$$

We accept $T_p^\circ = T / p$ as the value for $T_p^\circ \equiv (\partial T^\circ / \partial p)_s$, this is justified by experiments for most fluids [7]. In addition, we shall assume for simplicity that $\mu_\rho = 0$, i.e. disregard the magnetostriction effect, as is sometimes done for the liquid magnetic substances. We note that the sound waves in ferromagnetic fluids were also discussed in [8, 9].

When $M = K_0(\theta - T)$, we have the following expressions for a saturated liquid magnetic subsatnce

$$N^{-1} = 1 - \frac{4\pi K_0^2 T}{\rho c_p}\left(1 + \frac{\gamma_1 T}{\gamma_2 p}\right), \qquad m^{-1} = 4\pi\rho\mu^2$$

$$\rho x_{31} = a_s^2\left(1 - \frac{K_0 H T}{\rho p \gamma_2 a_s^2}\right), \qquad x_{23} = \frac{N}{\rho}\left(\frac{4\pi K_0^2 T}{p\gamma_2} - K_0 H\right)$$

$$x_{32} = -\frac{K_0 H T}{\rho c_p}\left(1 + \frac{\gamma_1 T}{\gamma_2 p}\right)$$

Let us assign the quantities appearing in these formulas, the following orders of magnitude which are characteristic for liquids under normal conditions:

$$\gamma_1 = 10^{-3}\,/\text{deg}, \quad \gamma_2 = 10^{-4}\,/\text{atm}, \quad K_0 = 3 \times 10^{-2}\ \text{gauss}/\text{deg}$$

$$T = 3 \times 10^2\ \text{deg}, \quad c_p = 10^{10}\ \text{erg}/\text{g} \times \text{deg}, \quad a_s = 10^5\ \text{cm}/\text{s}$$

$$\rho = 1\ \text{g}/\text{cm}^3, \quad p = 10^6\ \text{dyn}/\text{cm}^2 = 1\ \text{atm}$$

Then, according to estimates, the term ρx_{31} makes the greatest contribution to (7). Consequently, disregarding terms of the order higher than 10^{-6}, we obtain

$$\lambda_{4,5} = \pm c\varepsilon^{-1/2}, \qquad \lambda_{6,7} = \pm a_s\left(1 - \frac{K_0 H T}{p\gamma_2\rho a_s^2}\right)^{1/2}$$

which shows that in the example in question instability can occur in the fields $H > p\gamma_2\rho a_5^2/K_0T \sim 10^5$ gauss.

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PLANE ELECTROHYDRODYNAMIC FLOW WITH REVERSE CURRENT

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We consider the modes of flow of unipolarly charged jets in the case when the charged particles return along the peripheral zones of the hydrodynamic streams to the electrode-"emitter" under the action of both the induced and the external electic field. It is shown that the reverse current increases with increasing width of the section through which the charged particles enter, the electric charge density at this section and the intensity of the external retarding field. The electric current from the electrode emitter is reduced by the reverse current and the retarding effect of the external field.

During the present investigation we solve the two-dimensional electrohydrodynamic equations numerically, determine the local electrical parameters and the character of the flow established over the whole region.

The reserve currents result from the spatial (two-dimensional) character of the real electrohydrodynamic flows. The effect becomes of interest when constructing electrohydrodynamic energy transducers where the losses caused by